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1. A Lorentz transformation from inertial frame O, x^α to O' is $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$, where the transformation matrix $\Lambda^\alpha{}_\beta$ is

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

a) Find the velocity (speed and direction) of O' relative to O.

Solution. The off-diagonal entries determine $\beta\gamma$, that is $\frac{v}{c} = \frac{\beta\gamma}{\gamma} = \frac{0.75}{1.25} = 0.6$.

b) Find the transformation matrix $\Lambda_\alpha{}^\beta$ for $x'_\alpha = \Lambda_\alpha{}^\beta x_\beta$, where $x_\alpha = g_{\alpha\beta} x^\beta$.

Solution. The transformation matrix is the same as taking the matrix product $g_{\alpha\gamma}\Lambda^\gamma{}_\delta g^{\delta\beta}$. The matrix product of which is

$$\Lambda_\alpha{}^\beta x_\beta = \begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

2. Among the following expressions of coordinates x^α, which are scalar with respect to Lorentz transformation?

$$\begin{aligned} &x^0x^0 - x^1x^1 - x^2x^2 - x^3x^3, \\ &x^0x_0 - x^1x_1 - x^2x_2 - x^3x_3, \\ &x^0x^0 + x^1x^1 + x^2x^2 + x^3x^3, \\ &x^0x_0 + x^1x_1 + x^2x_2 + x^3x_3 \end{aligned}$$

where $x_\alpha = g_{\alpha\beta}x^\beta$.

Solution. The first and fourth equations above are invariant. The first is invariant since $ds' = ds$, where ds is the proper time. See definition of metric tensor for the fourth.

3. Show that the determinant of $g_{\mu\nu}$ is Lorentz scalar.

Solution. Compare the relation $g'_{\nu\mu} = \Lambda^\alpha \nu \Lambda^\beta \mu g_{\alpha\beta}$. Since the determinant is a linear operation, it follows that

$$\begin{aligned} g'_{\nu\mu} &= \Lambda^\alpha \nu \Lambda^\beta \mu g_{\alpha\beta} \\ \Longrightarrow ||g'_{\nu\mu}|| &= ||\Lambda^\alpha \nu \Lambda^\beta \mu g_{\alpha\beta}|| \\ &= ||\Lambda^\alpha \nu|| ||\Lambda^\beta \mu|| ||g_{\alpha\beta}|| \\ &= ||g_{\alpha\beta}|| \end{aligned}$$

since the determinant of a Lorentz transformation matrix is ± 1.

4. Show that the 4-acceleration $\frac{dU^\alpha}{ds}$, where U^α is 4-velocity, has only 3 independent components.

   Solution. First consider the relationship, $U^\alpha U_\alpha = 1$. Taking one derivative makes the subsequent dot product 0. Differentiate wrt s to get,

$$\begin{aligned} \frac{dU^\alpha}{ds}U_\alpha + U^\alpha \frac{dU_\alpha}{ds} &= 0 \\ \implies 2\frac{dU^\alpha}{ds}U_\alpha &= 0. \end{aligned}$$

   which implies that one component is a linear combination of the other three.

   Therefore, three components of the derivative are independent.

5. Consider the following reaction (π^0 - production)

$$\gamma + p \rightarrow p + \pi^0.$$

   The rest energy is $m_p c^2 = 938$ MeV for proton, and $m_{\pi^0} c^2 = 135$ MeV for the neutral pion π^0. If the initial proton is at rest in the laboratory, find the laboratory threshold gamma-ray (γ) energy for this reaction to work.

   Solution. ${P_\gamma}^\alpha$ of the photon is $(\frac{E_\gamma}{c}, \frac{E_\gamma}{c}, 0, 0)$ and ${P_p}^\alpha$ of the proton (initially at rest) is $(m_p c, 0, 0, 0)$. Look at the scalar quantity

$$({P_\gamma}^\alpha + {P_p}^\alpha)(P_{\alpha_\gamma} + P_{\alpha_p}) = \left(\frac{E_\gamma}{c} + m_p c\right)^2 - \left(\frac{E_\gamma}{c}\right)^2.$$

   Since this quantity is Lorentz scalar, the quantity will be the same in the center-of-mass frame. In the center-of-mass frame the minimum energy-momentum of proton and π^0 is $(m_p c + m_{\pi^0} c, 0, 0, 0)$ which results in the relation

$$\left(\frac{E_\gamma}{c} + m_p c\right)^2 - \left(\frac{E_\gamma}{c}\right)^2 = (m_p + m_{\pi^0})^2 c^2.$$

   Solve for E_γ,

$$E_\gamma = \frac{m_{\pi^0}^2 + 2m_{\pi^0} m_p)c^4}{2m_p c^2} \approx 144.7 \text{ MeV}.$$